Available online at www.sciencedirect.com

SciVerse ScienceDirect

IERI Procedia 3 (2012) 198 – 205

Procedia IERI

www.elsevier.com/locate/procedia

2012 International Conference on Mechanical and Electronic Engineering

Maximum Principle and Application of Parabolic Partial Differential Equations

Dong Fangliang

Hebi College of Vocation and Technology, Hebi, China

Abstract

We use the maximum principle in comparison principle and blowing up questions. In the last, we discuss the maximum principle of elliptic partial differential equation. By importing the strong maximum principle and Hopf lemma of boundary, and presents two results which are unknown before .And applied maximum to the problem of Poisson and the minimal surface equation.

Keyword: Maximum principle, comparison principle, blow up

1. Introduction

Parabolic partial differential equations partial differential equation is a typical one, its maximum principle in the whole differential system which occupies a very important position.

2. The basic theory of partial differential equations

In order to facilitate the introduction of maximum principle and narrative, this section first gives some definitions and notations.

Definition 1 (k-order partial differential equations) on a two or more variables with unknown function and the partial derivative of the equation is called partial differential equations; denoted. If the equation appears as the most higher-order derivatives of order k is called k-order partial differential equations.

Form: $F(D^k u(x), D^{k-1}u(x), \cdots, Du(x), u(x), x) = 0 \ (x \in R^n)$ is the k-order PDE.

Definition 2 (linear partial differential equations): the unknown function and its derivative are linear partial differential equations known as the linear partial differential equations.

The form: $\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha u = f(x) \quad a_\alpha(x)$ This is a given function.

Definition 3 (fully nonlinear partial differential) equations with highest-order derivative of the nonlinear partial differential equation known as the fully nonlinear partial differential equations.

Definition 4 (half-linear partial differential equations) on the derivative of the most advanced linear and its coefficient is only a function of the independent variables of partial differential equations known as the semi-linear partial differential equations.

The form: $\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + F(D^{k-1}u, \cdots, Du, u, x) = 0$

Definition 5 (quasi-linear partial differential equations) on the derivative of the most advanced linear, but the coefficient of the derivative may contain low-level quasi-linear partial differential equations known as partial differential equations.

The form: $\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \cdots, Du, u, x) D^\alpha u + F(D^{k-1}u, \cdots, Du, u, x) = 0$

Which fully nonlinear partial differential equations, semi-linear partial differential equations and quasi-linear partial differential equations referred to as non-linear partial differential equations.

In the second-order linear partial differential equations of general form:

$$L[u] \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + hu = f \tag{1}$$

Which are functions, now the second-order general equation (1) for classification? Based representation of the point, said point value in the matrix. Are classified as follows:

Defined if the point is positive definite or negative definite, then the equation at the point known as the oval. If the characteristic values in addition to a 0 with the remaining eigenvalues are positive or negative, equation at the point called parabolic. If none of the eigenvalues of 0 and have the same number of eigenvalues, the equation is called hyperbolic.

Define (uniformly parabolic type operator) equation (1) is called parabolic zone: if every point is in the parabolic type. In the same parabolic, if for each point are parabolic and there is a normal number of eigenvalues at the same time making a meet or while meeting all the set up.

3. Maximum principle and comparison principle

$$(L+h)[u] = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + hu - \frac{\partial u}{\partial t} = f \qquad t \in (0,T] \tag{2}$$

Which is a function called homogeneous if the heat transfer equation, or non-homogeneous heat conduction equation.

The following theorem gives a non-homogeneous heat conduction equation of the general maximum principle and the strong maximum principle.

Theorem: (heat conduction equation of the non-negative maximum principle)

If the equation (2) solution, and is consistent with the parabolic and is

The maximum value will be non-negative on the parabolic boundary in reach, that is

If there is simply connected and so, there

Usually called the principle of maximum heat conduction equation for heat conduction equation the strong maximum principle.

From the physical point of view, if the object is not within the "heat" in the heat transfer process, the temperature always tends to equilibrium, the maximum temperature at the heat transfer to ambient temperature at the lowest temperatures tend to rise, so the maximum temperature of the object and minimum temperature is always at the initial time or the object boundary to achieve. Physical mathematical description of this phenomenon is the so-called "maximum principle."

Theorem 1 gives a general maximum principle, but for a specific problem may be derived from the maximum principle has its unique place, we might look at the problem and gives the maximum principle to prove.

Theorem 2 (Maximum Principle question) if the following problem is solved,

$$\begin{cases} a^2\Delta u - u_t = 0 & (x,t) \in R^n \times (0,T) \\ u = g(x) & (x,t) \in R^n \times \{t=0\} \end{cases} \tag{3}$$

But also to meet the constant and greater than 0. There

Prove () first assume; then there exists a meet

$$4b(T+\varepsilon) < 1 \tag{4}$$

Fixed $x_0 \in R^n$ and define the function w, $w(x,t) = u(x,t) - \frac{\lambda}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-x_0|^2}{4(T+\varepsilon-t)}}$ t> 0

Which functions as $\frac{\lambda}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-x_0|^2}{4(T+\varepsilon-t)}}$ the basis for solution of heat conduction equation, it is clear that (3), the homogeneous equation. Properly select and set a positive number r such heat conduction equation based on a maximum principle $\max\{w(x,t) \mid (x,t) \in \overline{U_T}\} = \max\{w(x,t) \mid (x,t) \in \Gamma_T\}$

If, $x \in R^n, t = 0$ when,

$$w(x,0) = u(x,0) - \frac{\lambda}{(T+\varepsilon)^{n/2}} e^{\frac{|x-x_0|^2}{4(T+\varepsilon)}} \le u(x,0) = g(x) \tag{5}$$

When $x \in R^n, 0 < t \le T, |x - x_0| = r$ the time

$$w(x,t) = u(x,t) - \frac{\lambda}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \le L\,\mathrm{e}^{b(|x_0|+r)^2} - \frac{\lambda}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}}.$$

By (4), $\frac{1}{4(T+\varepsilon)} = b + \kappa \ (\kappa > 0)$ we know into calculated

$$\le L\,\mathrm{e}^{b(|x_0|+r)^2} - \lambda\{4(b+\kappa)\}^{\frac{n}{2}} e^{\frac{r^2}{4(T+\varepsilon)}} \le \sup\{g(x) \mid x \in R^n\} \tag{6}$$

Where, r to make the selection (6) holds. Integrated (5) (6) and let a two-type

$$\sup\{u(x,t) \mid (x,t) \in \overline{U_T}\} = \sup\{g(x) \mid x \in R^n\}$$

(b) When (4) does not hold, as long as the interval broken down into small, can always be found making holds.

Also in the differential equation in position with the maximum principle, another principle is more important principle, the following maximum principle is given with one type of parabolic comparison principle to prove.

Theorem (a class of reaction-diffusion equations of the comparison principle) is defined

$$Pu \equiv Pu(x,t) = \frac{\partial u}{\partial t} - \Delta u = a(x)u + b(x)(\nabla u) + c(x)\int_{t0}^{t} u\,dt \quad , x \in U, 0 \le t \le T \tag{7}$$

Which is a,b,c bounded on the bounded domain function. If there are two functions to meet: $u,w \in C^{2,1}(U_T) \bigcap C(\overline{U_T})$

(i) $Pu(x,t) \le Pw(x,t) \quad (x,t) \in U_T$

(ii) Initial boundary $u(X,t)$, $\le w(x,t)$ the initial boundary is expressed as $((0,T] \times \partial U) \bigcup (\{0\} \times \overline{U})$

There conclusions: $u(x,t) \le w(x,t)$ $(x,t) \in \overline{U_T}$

That because a,b,c of the bounded domain bounded function, so

$A = \sup_{x\in U} |a(x)|, B = \sup_{x\in U} |b(x)|, C = \sup_{x\in U} |c(x)|$

$$|c(x)\int_{t0}^{t} u_2 dt - c(x)\int_{t0}^{t} u_1 dt|_{L^\infty(\Omega)} \le |c(x)| \times T \times |u_1 - u_2|) \le CT|u_1 - u_2| \tag{8}$$

$$|\{a(x)u_1 + b(x)(\nabla u_1)\} - \{a(x)u_2 + b(x)(\nabla u_2)\}|_{L^\infty(\Omega)} \le A|u_1 - u_2| + B|\nabla u_1 - \nabla u_2| \tag{9}$$

Let and $v = (u - w)$ $L = \max\{A, B, C \times T\}$ put in P ,we can obtain

$$\frac{\partial v}{\partial t} - \Delta u = Pu(x,t) - Pw(x,t)$$

$$= \{a(x)u + b(x)(\nabla u)\} - \{a(x)w + b(x)(\nabla w)\} + \{c(x)\int_{t0}^{t} u\,dt - c(x)\int_{t0}^{t} w\,dt\}$$

$$\le L\{|u-w| + |\nabla u - \nabla w| + |u-w|\}_{L^\infty(\Omega)}$$

$$\le L\{|v| + |\nabla v| + |v|_{L^\infty(\Omega)}\}.$$

Let $\alpha > 2L + |\nabla v|_{L^\infty(\Omega)})$,then $z(x,t) = v(x,t)e^{-\alpha t}$ meet

$$\frac{\partial z}{\partial t} - \Delta z \le L \times \{|z| + |\nabla z| + |z|_{L^\infty(\Omega)}\} - \alpha \times z(x,t) \tag{10}$$

And substituted into the operator to make available

Order $v = (u - w)$, then $L = \max\{A, B, C \times T\}$ meet the following formula

If negative, including the maximum point of non-contradictory set

Therefore, there is no non-negative maximum value that the conclusion holds

Note: The partial differential equation and is the deformation of the two curves, and the first curve strict in the second curve, then the distance between the two curves satisfy the maximum principle.

4. Parabolic partial differential equations of the application of maximum principle

The theory of partial differential equations the maximum principle is a basic and important conclusions. This principle may help to prove the uniqueness of the solution of differential equations, and other properties.

Theorem 4 (on a bounded domain uniqueness of the solution) or $g(x,t) \in C(\Gamma_T), f(x,t) \in C(U_T)$ set the following initial boundary value problem at most one solution. $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$

$$\begin{cases} a^2 \Delta u - u_t = f(x,t) & (x,t) \in U \times (0,T) \\ u = g(x,t) & (x,t) \in \partial U \times [0,T] \end{cases} \tag{11}$$

Proof let the two solutions meet, so that was substituted into equation

$$\begin{cases} a^2 \Delta w - w_t = 0 & (x,t) \in U \times (0,T) \\ w = 0 & (x,t) \in \partial U \times [0,T] \end{cases},$$

Using Theorem 1 we can see $w = 0$, the conclusion is proved.

Theorem 5 (Cauchy problem of uniqueness of solution) $g(x) \in C(\Gamma_T), f(x,t) \in C(U_T)$ is located , is $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$ at most one solution to meet the Cauchy problem

$$\begin{cases} a^2 \Delta u - u_t = f(x,t) & (x,t) \in R^n \times (0,T) \\ u = g(x) & (x,t) \in R^n \times \{t = 0\} \end{cases} \tag{12}$$

And there are $u(x,t) \le Le^{bx^2} (x \in R^n, 0 \le t \le T), L$ constant and greater than 0.

Proof $u_1(x,t), u_2(x,t)$ let the (3.10) two solutions meet, so that was substituted into equation

$$\begin{cases} a^2 \Delta w - w_t = 0 & (x,t) \in R^n \times (0,T) \\ w = 0 & (x,t) \in R^n \times \{t = 0\} \end{cases}$$

Using Theorem 2 we can see, the conclusion is proved

At this stage it is the most studied aspects combining extreme maximum principle applied to functional problems and blasting, following the bursting of the maximum principle applied to the issue of a little discussion.

5. Maximum Principle for blasting problems

To illustrate the bursting phenomenon, we first introduce the definition of burst, and then give an example to illustrate.

Definition 8 (blasting) a time parameter of the solution containing a limited period of time lost in the formal nature of resulting singularity, that the solution itself or some derivative tends to infinity in finite time, a phenomenon known as the solution of the blast.

Example 1 is non-linear heat conduction equation of the mixed initial value - the problem

$$\begin{cases} \frac{\partial u}{\partial t} \text{-} \vartriangle u = u^2 & (13) \\ \frac{\partial u}{\partial n} | z = 0 & (14) \\ u(\mathrm{x},0) = \mathrm{x}^2 & (15) \end{cases}$$

Solution region is cylindrical region, the lateral boundary, and for a bounded region, and appropriate for its smooth boundary. The mixed boundary problem that was not there in the whole of the classic solution, which occurs in solution within a limited time blasting (blow up)

To prove this, we introduced in [6] in a theorem. That

Theorem $u \in C^3(U) \cap C^2(\overline{U})$ let be the following heat conduction equation of the boundary problem is solved,

$$\begin{cases} \frac{\partial u}{\partial t} \text{-} \vartriangle u = f(\mathrm{x},u) & (\mathrm{x},t) \in \mathrm{U} \times (0,\mathrm{T}), & (16) \\ \frac{\partial u}{\partial n} | z = 0, & (\mathrm{x},t) \in \partial\mathrm{U} \times (0,\mathrm{T}), & (17) \\ u(\mathrm{x},0) = u_0(\mathrm{x}) \geq 0, & (\mathrm{x},t) \in \mathrm{U} \times (t=0) & (18) \end{cases}$$

When the following conditions are true,

(i) $f(x,u) \geq 0$,

(ii) $\Delta_x f - |\nabla_x (\frac{\partial f}{\partial u})|^2 \geq 0$

(iii) $\frac{\partial^2 f}{\partial u^2} \geq 1$,

(iv) $\int_{M0}^{+\infty} \frac{1}{f(x_0,s)} ds < +\infty$, Which $M_0 = \max u_0(x)$,

Therefore, to meet the conditions of Theorem 6, it occurs in a limited time domain solution of blasting

Note: In fact, Example 1 can also be used to prove the following methods.

Prove that the equation (13) can be integrated at both ends of the
The use of divergence theorem and noting the boundary conditions (14) are:

$$\int_U \Delta u dx = \oint_\Sigma \frac{\partial u}{\partial n} dS = \oint_\Sigma 0 \times dS = 0$$

So you can get

$$\frac{d}{dt} U(t) = \int_U u^2(x,t) dx \tag{19}$$

Note the use ***Hölder*** of inequality are:

$$U(t) = \int_U u(x,t) dx \le (\int_U u^2(x,t) dx)^{\frac{1}{2}} \times |\Omega|^{\frac{1}{2}},$$

Which $|\Omega|$ indicates the size of the region, we can get into

$$\frac{dU(t)}{dt} \ge \frac{1}{|\Omega|} \times U^2(t) \tag{20}$$

And $U(0) = \int_\Omega x^2 dx > 0$, if we denote the following equation for the solution of the problem

$$\begin{cases} \dfrac{dV(t)}{dt} = \dfrac{1}{|\Omega|} \times V^2(t) \\ V(0) = \int_\Omega \varphi(x) dx > 0 \end{cases} \tag{21}$$

Clearly a solution $U(t) \ge V(t)$, but the problem solvable

$$V(t) = \frac{V(0)}{1 - V(0)t},$$

One time there was, that the solution to rupture, and only in the time interval $[0, \frac{1}{V(0)})$ on the local solution.

So $U(t)$ will a limited period of time tends to infinity, so the original mixed solution of the problem must burst in finite time.

References

[1] Murray H. Potter & Hans F. Weinberger. Maximum Principle in Differential Equations. New Jersey: Prentice-Hall, 1967
[2] D. Gilberts & N.S Trudging. Elliptic Partial Differential Equations of Second Order. Springer-velar, New York, 1977
[3] Lawrence C. Evans. Partial Differential Equations. Rhode Island, 1998. .
[4] Philippe Soup let. Blow-up in No local Reaction-Diffusion Equations。Society for Industrial and Applied Mathematics,1998,29 (8):1301-1334
[5] Philippe Soupiest. Recent Result and Problems on Parabolic Equations with Gradient Nonlinearities. Elect ionic Journal of Differential Equations.,2001(20):1-19
[6] YU J., Zhao Y, Weighted Approximation of Functions with Singularity by q-Baskakov Operators, IEIT Journal of Adaptive & Dynamic Computing, 2012(2), Apr 2012, pp:5-11. DOI=10.5813/www.ieit-web.org/IJADC/2012.2.2
[7] Zhao C.H., Zhang J., Zhong X.Y., Chen S.J., Liu X.M, Analysis of Tower Crane Monitoring and Life Prediction, IEIT Journal of Adaptive & Dynamic Computing, 2012(2), Apr 2012, pp:12-16. DOI=10.5813/www.ieit-web.org/IJADC/2012.2.3

[8] Zhao C.H., Chen S.J., Liu X.M., Zhang J., Zeng J, Study on Modeling Methods of Flexible Body in ADAMS, IEIT Journal of Adaptive & Dynamic Computing, 2012(2), Apr 2012, pp:17-22. DOI=10.5813/www.ieit-web.org/IJADC/2012.2.4
[9] Chen G.Q., Jiang Z.S., Wu Y.Q, A New Approach for Numerical Manifold Method, IEIT Journal of Adaptive & Dynamic Computing, 2012(2), Apr 2012, pp:23-34. DOI=10.5813/www.ieit-web.org/IJADC/2012.2.5